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A SIMPLE PROCEDURE FOR PERFORMING SECOND ORDER ANALYSIS USING A LINEAR STRUCTURAL ANALYSIS PROGRAM

SECOND ORDER ANALYSIS

A Simple Procedure for Performing Second Order Analysis Using a Linear Structural Analysis Program

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Abstract

This paper describes an iterative procedure for performing second order geometric non-linear analysis using a basic linear structural program. The advantage of this procedure is that it can be applied by a user of the computer program without having to modify the internal workings of the program. Therefore, non-linear analysis is now within reach of any engineer using a linear finite element program. This iterative procedure can also be used to predict the critical buckling load quickly without having to perform repeated load increments and without having to assume the effective buckling length.

Keywords: Buckling, second order analysis, structural analysis

1. Introduction

Linear Finite Element programs perform a first order analysis assuming that the properties of the structure are constant and that deflections do not have any effect on the structural behaviour. However, in some cases, the deflection of the structure can have a goemetric second order effect on the behaviour of the structure, which is not captured by the linear first order analysis.

This type of geometric non-linearity can be analysed with a linear structural analysis program using an iterative procedure as described in this paper.

2. Second Order Deflection

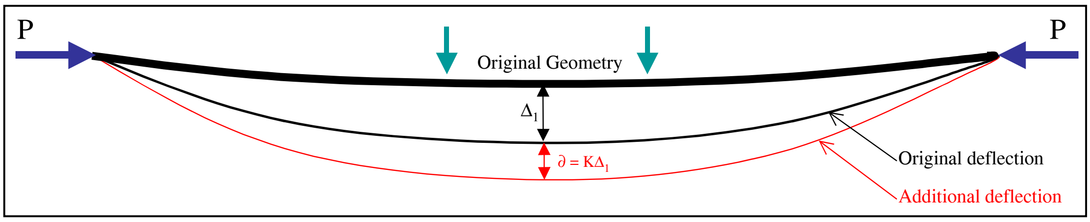


Fig. 1 - Second Order Deflection

In a normal linear computer analysis, the output deflected shape is different from the input geometry (fig.1). The structural loads acting on this difference is what causes an additional 2nd order deflection. This additional deflection ∂ is proportional to the axial load and the original deflection Δ_1 :-

$$\partial = kP\Delta_1 = K\Delta_1 \text{..... eqn. (1)}$$

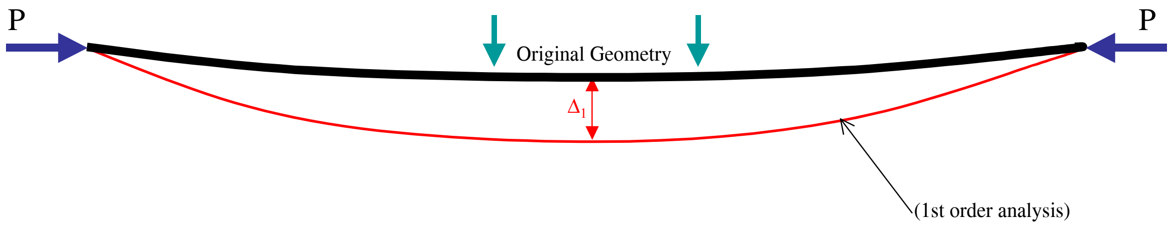
We can use the computer to calculate this additional deflection ∂ by running the analysis again using the same loads acting on the deflected shape of the structure as the input. The second analysis gives an incremental deflection to the structure and thus a new deflected shape. This gives a better evaluation of the deflected shape of the structure, but may still contain unbalanced internal forces if the output shape is still not equal to the input shape.

We now run a third analysis using the same loads and the new deflected shape as the input. Again we have a different deflected shape. We notice that the difference between the new shape and the previous one is less than before. By repeatedly analyzing the structure using the revised deflected shape, we eventually reach a stage where the output deflected shape and matches the input shape. The iterative procedure has thus calculated the geometric second order deflection of the structure.

3. Analysis Procedure

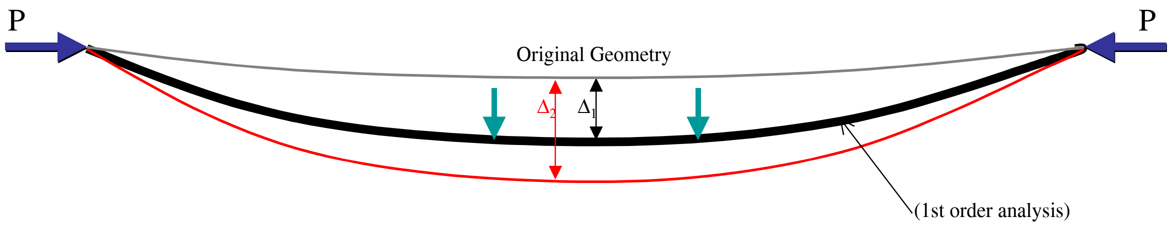
Implementation of the procedure is very simple. Essentially the procedure consists of simply running the analysis, adding the computed deflections to the original (undeflected) geometry, and re-running again until the computed deflections for successive runs converge. Computations can be easily done by cutting and pasting from the finite element program to a spreadsheet

1. First make a normal analysis which gives the first order linear deflection Δ_1.



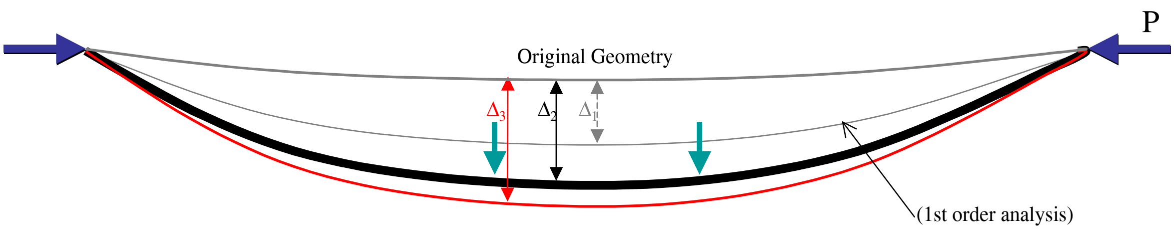
2. Then add Δ_1 to the original geometry and run again to get a new deflection Δ_2. This deflection is measured from the original geometry of the structure

$$\Delta_2 = \Delta_1 + K\Delta_1$$



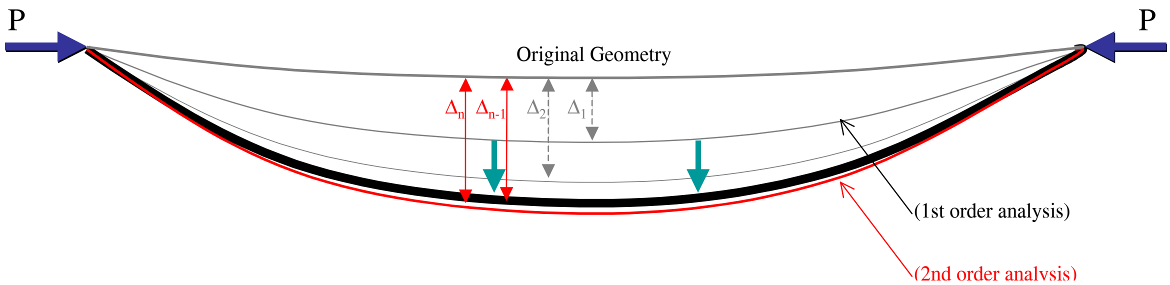
3. Add Δ_2 to the original geometry and run to get Δ_3. Again the deflection is measured from the original geometry.

$$\Delta_3 = \Delta_2 + K^2\Delta_1$$
$$= \Delta_1 + K\Delta_1 + K^2\Delta_1$$



4. Repeat until there is negligible change between deflections Δ_n and Δ_{n-1} for successive runs. The final Δ_n is the second order deflection taking into account the geometric non-linear effects.

$$\Delta_n = \Delta_{n-1} + K^{n-1}\Delta_1$$
$$= \Delta_1 (1 + K + K^2 + K^3 \ldots + K^{n-1})$$
$$\approx \Delta_1 / (1 - K)$$



4. Interpreting Results

This method captures the overall second order behaviour of the structure (P-BIG DELTA). The second order behaviour of individual members in between joints (P-small delta) is not captured. Therefore, to study individual members, they should be sub-divided into several shorter components.

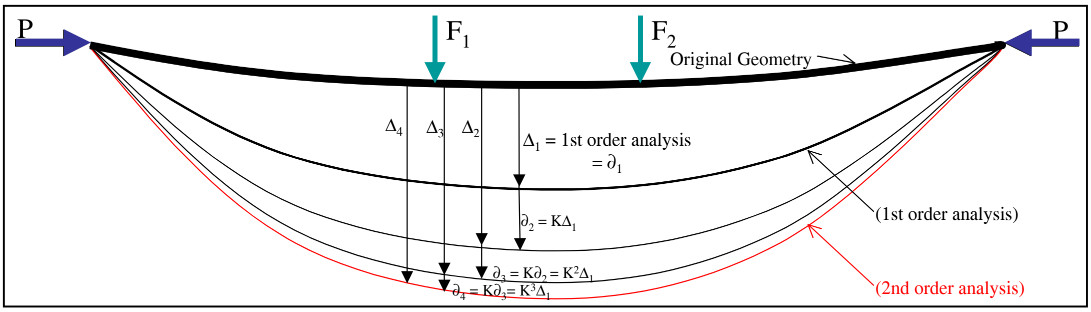


Fig. 2 – Successive Iterations Related by K

For successive iterations, the incremental deflections ∂_i and ∂_{i-1} are related by a ratio $K = \partial_i / \partial_{i-1}$. This ratio K can be considered as the geometric effect that the deflection of the structure has on the subsequent deflection. The value and rate of change of K with successive iterations gives information on the second order behaviour of a structure.

For pure buckling, the ratio K between incremental deflections for successive iterations is constant. If the initial deflection is Δ_1, then the subsequent incremental deflections are $K\Delta_1$, $K^2\Delta_1$, $K^3\Delta_1$, $K^4\Delta_1$ etc. (fig. 2). The magnitude of K determines the rate at which successive iterations will converge to the final value. Itf the structure is stable, K < 1. If K = 1, then the critical buckling load has been reached, and the 2nd order deflection goes to infinity.

The total deflection after n iterations is $\Delta_n = \Delta_1 (1 + K + K^2 + K^3 \ldots + K^{n-1})$ etc.
After many iterations, $\Delta_\infty \approx \Delta_1 / (1 - K)$
This is the same as predicted by classical beam-column theory:- $\Delta_\infty = \Delta_1 / (1 – P / Pcrit)$
Thus K = P / Pcrit. i.e. the proportional factor K is equal to the ratio of the applied axial load to the critical buckling load.

In fig. 3, the second order deflection is plotted against total deflection for successive iterations. The relationship between incremental deflections for successive iterations and the total deflection vs the first order deflection can be seen graphically.

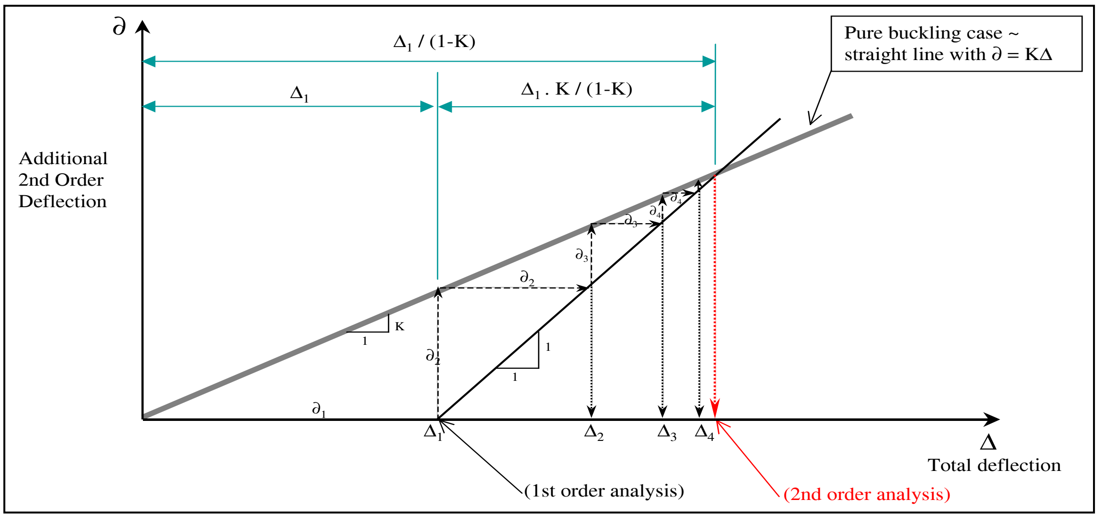


Fig. 3 – Plot of 2nd Order vs Total Deflection

5. Rigid Body Displacement

For many structures, the plot of the second order vs total deflection does not always cross the origin at zero. There appears to be an initial displacement D which does not influence the second order deflection. (Fig. 4)

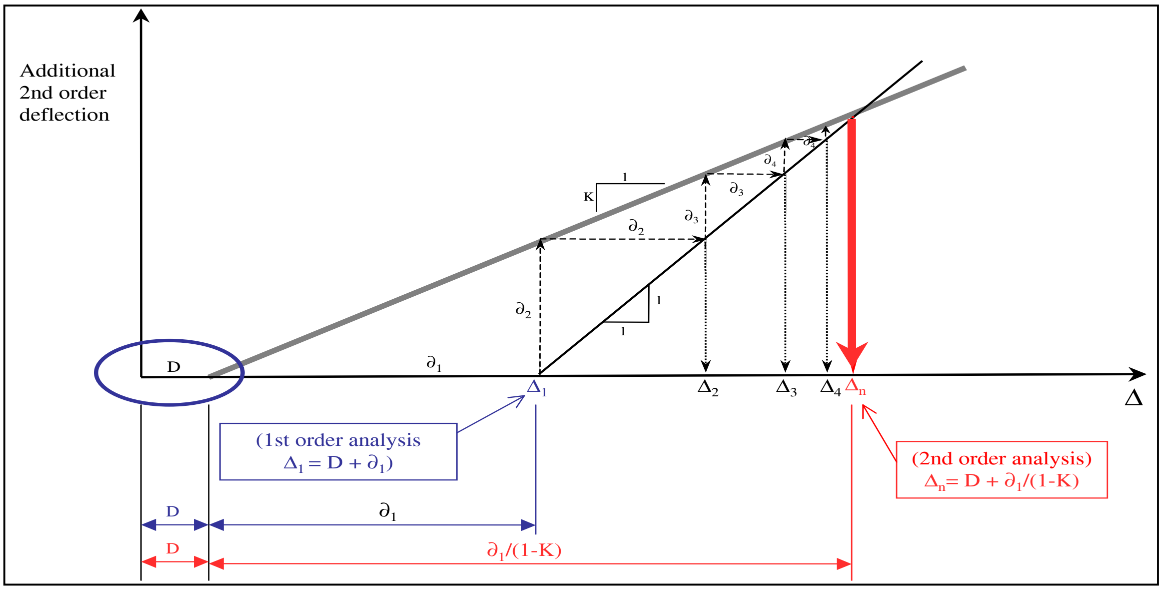


Fig. 4 – Rigid Body Displacement D

This rigid body displacement D can be visualized in the example of a beam column resting on spring supports. (Fig. 5) In this case, the entire beam moves laterally by a distance D, and this D has no effect on the second order behaviour of the system.

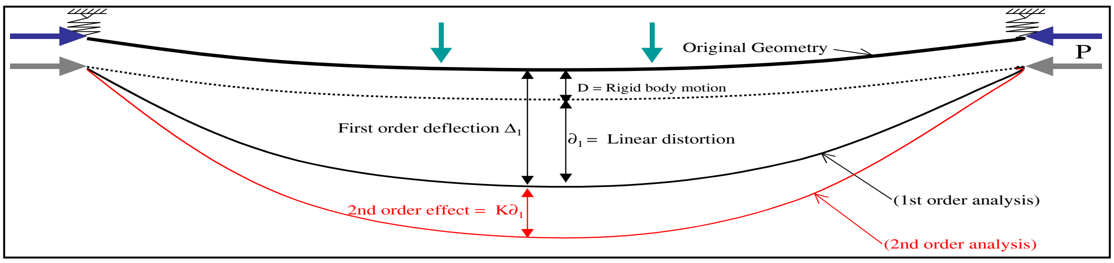


Fig. 5 – Rigid Body Displacement of Beam on Spring Supports

For lateral deflection of members under axial load, the value of K remains constant after adjustments are made for the rigid body displacement D. There are other modes of 2^{nd} order behaviour such as the effect of lateral loads on axial deflections where K may not be constant. By studying the behaviour of the multiplier K over successive iterations, much information can be gleaned about the 2nd order non-linear behaviour of the structure. This can be a scope for interesting further research on second order behaviour of structures.

6. Prediction Of Buckling Load

The ratio K between incremental deflections for successive iterations can be used to predict the critical axial buckling load Pcrit on the structure. Since K = P / Pcrit, the critical buckling load Pcrit = P / K. Therefore, it is possible to directly calculate the critical buckling load from the applied load P once K is obtained. It is not necessary to increment the applied load P to get the buckling load, since K is constant for pure buckling. There is also no need to assume an effective buckling length, as the critical buckling load Pcrit is computed from the applied load and K.

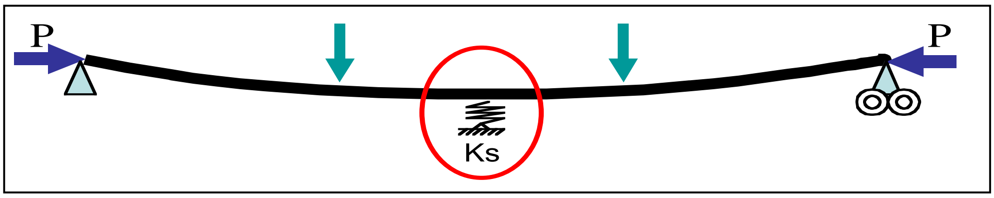


Fig. 6 – Axially Loaded Beam with Central Spring Support

Fig. 6 shows an analysis of an axially loaded beam with a central spring support. By varying the spring stiffness, the critical axial buckling load will be changed. Using this procedure, one second-order analysis was made for each value of spring stiffness to calculate the critical buckling loads.

The results are normalized against the Euler buckling load $Pe = \pi^2 EI / L^2$ and plotted in order to compare with the theoretical analysis found in Timoshenko[2] (fig. 7).

EFFECT OF CENTRE SPRING STIFFNESS

Ks	K	1/1-K	Pcrit = P/K	Ks*L/Pe	Pcrit/Pe
0	0.589	2.43	169.8	0.0	1
0.05	0.369	1.58	271.2	2.9	1.60
0.07	0.321	1.47	311.2	4.1	1.83
0.1	0.270	1.37	370.3	5.9	2.18
0.15	0.214	1.27	466.8	8.8	2.75
0.25	0.155	1.18	644.3	14.7	3.80

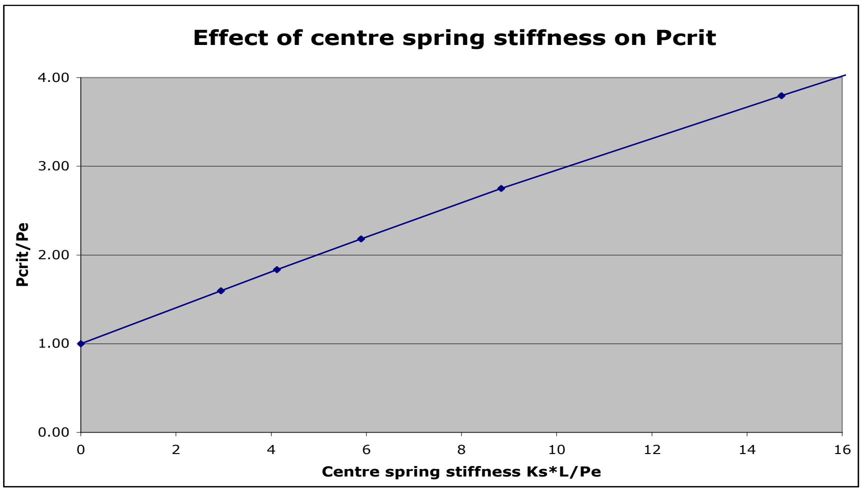


Fig. 7 – Comparison with Fig. 2-26 of Timoshenko's "Theory of Elastic Stability"

There is a good correlation between the results of the analysis and Timoshenko's theoretical calculations.

7. Skewed Arch Roof Structure

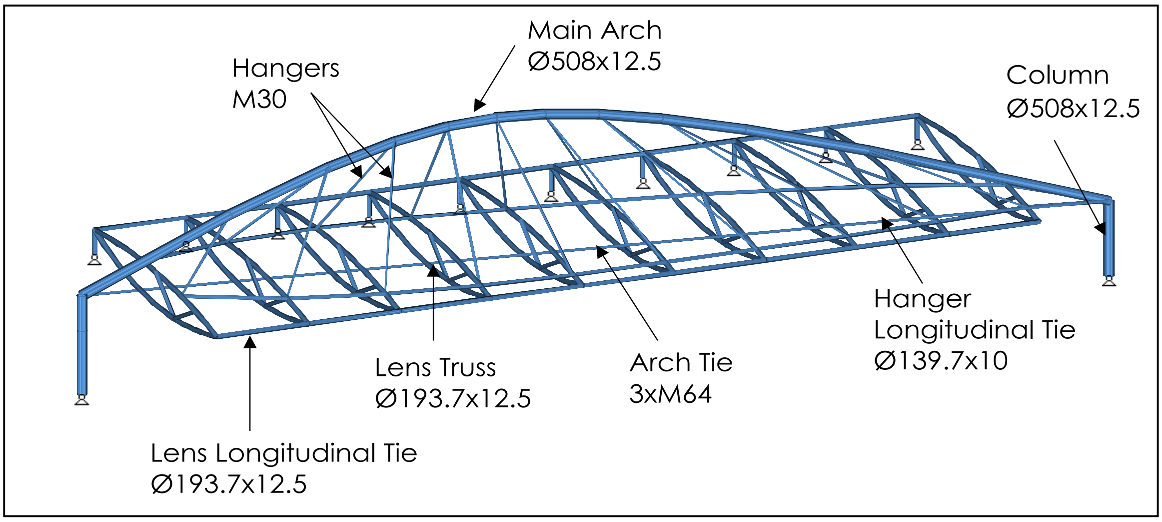
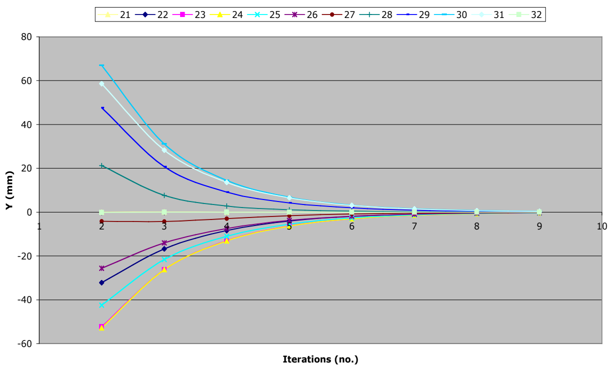
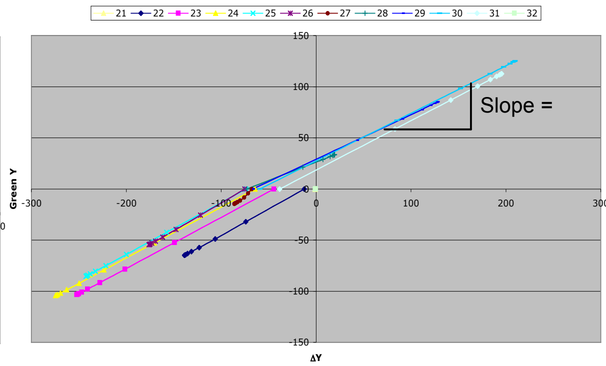


Fig. 8 – Skewed Arch Roof Structure

A second order analysis was made on the roof structure shown in fig. 8 using the iteration method described and the results compared with an independent checker's analysis using SAP2000 as well as an in-house analysis using Multiframe4D non-linear. The analysis results are similar with all three programs. The iterative procedure gives very similar results to the conventional Newton-Raphson method, and can be used for complicated structures as well as simple beam-columns.

This structure demonstrates a non-linear behaviour that is quite different from the linear one. The 2^{nd} order analysis converges quite rapidly as shown in fig. 9. The movements predicted by non-linear analysis correlate well with the movements observed in a physical scale model constructed from wires and brass tubing. K will vary at different points on the structure, depending on the second order effect of a particular loading on the point on the structure (fig. 10). Certain parts of a structure with larger values of K will be more prone to buckling than others.





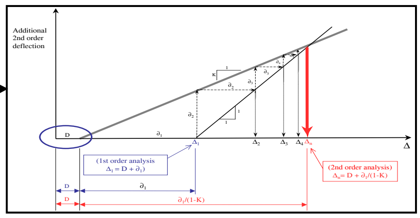
The plot of incremental deflection at each iteration step shows a pattern of convergence after only a few iterations. The different curves are for different locations on

The plot of 2nd order deflection vs total deflection shows a linear relation with a slope of K=P/Pcrit. This K can be used to predict the buckling load at each point on the

Fig. 9 – Incremental Deflections

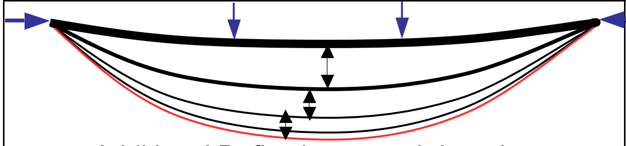
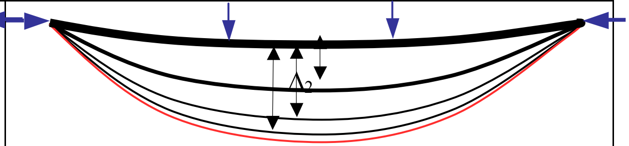
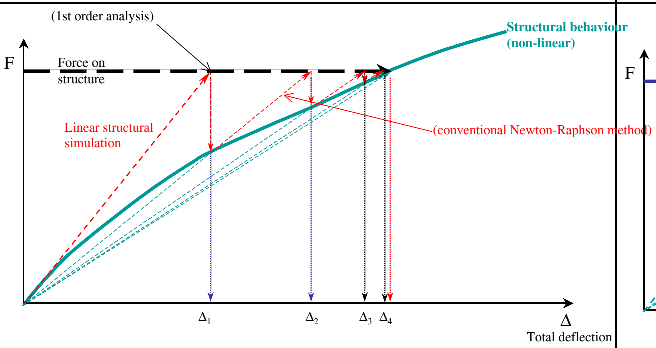
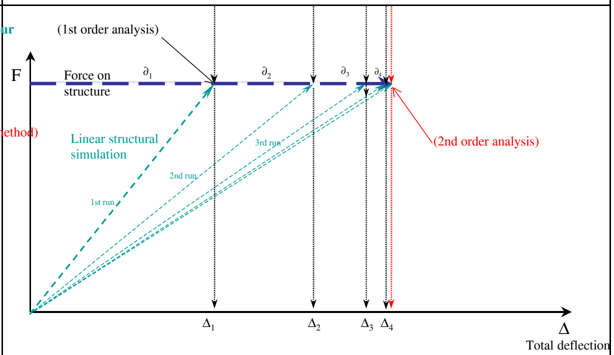
Fig. 10 – 2^{nd} Order vs Total Deflections

Compare fig. 10 with fig. 4 on the right



8. Comparison With Conventional Newton-Raphson Iteration

A common algorithm for calculating second-order behaviour in commercially available structural analysis programs uses the Newton-Raphson type of iteration. The table below compares the new method with the conventional method.

Conventional Method	**This Method**
 Additional Deflection at each Iteration	 Total Deflection at each Iteration
Deflections of the structure cause an imbalance in the internal forces	Structural forces acting on deflections of the structure cause additional (2nd order) deflections on the structure
The unbalanced forces are applied on the deflected structure to calculate additional (2nd order) deflections	The total deflection (including the additional deflection) is calculated by re-analyzing the structure with the deflected geometry
This is repeated until the additional deflections converge to zero.	This is repeated until the computed total deflected shape converges with the input geometry.
Manipulation of the computer program's internal matrices is required. Therefore, the method cannot be implemented by a normal user of the linear analysis program.	No internal manipulation of the structural forces or matrices is necessary. All that is required is to adjust the input geometry of the subsequent runs. Therefore, it can be done by an ordinary user of a linear finite element program without having to revise the internal workings of the program itself.
 Red dotted line shows force vs deflection for successive iterations with conventional method	 Green dotted line shows force vs deflection for successive iterations with this method

9. Conclusions

A method has been developed for performing second order geometric non-linear analysis using a basic linear structural program. This puts non-linear analysis within reach of any engineer with access to a linear structural analysis program.

- This method employs successive runs of the linear analysis using the same loads acting on the deflected geometry of the structure. It works on the principle that the loads acting on the deflections of a structure cause additional (2nd-order) deflections that can be computed by re-analyzing the structure with the deflected geometry.

- The method accurately calculates the behaviour of a simple beam column on spring supports as predicted by buckling theory. It has also been used for complex structures and gives similar results to conventional programs which use the Newton-Raphson type of iteration method.

- As a by-product of this method, the rate of change of deflections computed for successive iterations can be used to predict the critical buckling load on the structure without having to increment the applied load. This technique has been verified with the theoretical solution for simple buckling cases.

- The study of the incremental deflections between successive iterations can yield significant information about the non-linear behaviour of the structure and this is a scope for further research.

REFERENCES

1. McGuire, Gallagher, Ziemian: *"Matrix Structural Analysis",* Wiley & Sons, 2nd ed. 2000

2. Timoshenko & Gere: *"Theory of Elastic Stablility",* McGraw-Hill, 2nd ed. 1961

APPENDIX - Proof Cases

To validate the method, the following cases were run and the results compared with theory:-

Case 1. Beam-column buckling

For a steel pipe 168mm diameter x 5mm thick, 10 m long divided into 10 segments, the buckling load using this method was 170.3k and compares well with the calculated Euler buckling load of 168.9kN.

Case 2. Beam-column with spring supports at both ends

As above, but with spring supports to introduce rigid body translation into the system. Computed buckling load is the same as above. Lateral deflection under self weight and axial compression of 100kN is computed to be 46.0mm after 10 iterations, compared with the theoretical value of 46.6mm. Rigid body translation under self-weight is computed as 9.86mm vs theoretical 9.83mm.

Case 3. Tension case

As above, but with a tension load of 100kN. Convergence rate is the same and total lateral deflection is computed to be 19.25mm vs theoretical 19.28mm, with the same rigid body translation as above.

Case 4. Divergent tension

As above, but tension increased. At >170kN tension, which is equal in magnitude to the Euler buckling load, the computation failed to converge.

Case 5. Beam-column with centre spring support

Case 1 was run with a spring support at the mid point of the beam. Results compared with Timoshenko's theoretical analysis.

Case 6. Complex structure

The procedure was used on a complex skewed arch roof structure. Results for 2nd order analysis are similar to that obtained using commercial programs SAP2000 and Multiframe4D.